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Fixed point theorem in intuitionistic fuzzy metric space

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Abstract: In this paper, we generalize fuzzy metric space in term of fixed point theorem in Intuitionistic fuzzy metric space.

Key words: Fixed point, fuzzy metric space and Intuitionistic fuzzy metric space.

1. Introduction

Fuzzy set theory was first introduce by Zadeh [1] in 1965to describe the situation in which data are imprecise or vague or uncertain. Thereafter the concept of fuzzy set was generalized as intuitionistic fuzzy set by K. Atanassov [2] in 1984. Alaca et al. [5] using the idea of intuitionistic fuzzy sets, they defined the notion of intuitionistic fuzzy metric space as Park [4] with the help of continuous t-norms and continuous t-conorms. The concept of fuzzy metric space was first introduced by Kramosil and Michalek [3] but using the idea of Intuitionistic fuzzy set, It has a wide range of application in the field of population dynamics, chaos control, computer programming, medicine, etc.

Introducing the contraction mapping with the help of the membership function for fuzzy metric, several authors [6,7,8] established the Banach fixed point theorem in fuzzy metric space.

2. Preliminaries

We include some definitions which will be needed in the upcoming theorems

Definition 2.1

A binary operation $* : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is continuous t-norm if $*$ is satisfying the following conditions.

(1) $*$ is commutative and associative;

(2) $*$ is continuous;

(3) $a * 1 = a$ for all $a \in [0, 1]$;

(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 2.2

A binary operation $\lozenge : [0, 1] \times [0, 1] \to [0, 1]$ is continuous t-conorm if $\lozenge$ is satisfying the following conditions:

(1) $\lozenge$ is commutative and associative;

(2) $\lozenge$ is continuous;

(3) $a \lozenge 0 = a$ for all $a \in [0, 1]$;

(4) $a \lozenge b \leq c \lozenge d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 2.3

A 5-tuple $(X, M, N, *, \lozenge)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\lozenge$ is a continuous t-conorm and M, N are fuzzy sets on $X2 \times [0, \infty)$ satisfying the following conditions:

(1) $M(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$;

(2) $M(x, y, 0) = 0$ for all $x, y \in X$;

(3) $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;

(4) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(5) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $x, y \in X$ and $s, t > 0$;

(6) for all $x, y \in X$ and $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous;

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(7) $\lim_{n\to\infty} M(x, y, t) = 1$ for all $x, y \in X$

(8) $N(x, y, 0) = 1$ for all $x, y \in X$

(9) $N(x, y, t) = 0$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;

(10) $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(11) $N(x, y, t) \lozenge N(y, z, s) \geq N(x, z, t + s)$ for all $x, y \in X$ and $s, t > 0$;

(12) $\lim_{n\to\infty} N(x, y, t) = 0$ for all $x, y \in X$

(13) for all $x, y \in X$ and $N(x, y, .) : [0, \infty) \to [0, 1]$ is right continuous;

Then (M, N) is called an intuitionistic fuzzy metric space on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respective to t, respectively.

Definition 2.4

Let $(X, M, N, *, \lozenge)$ be an intuitionistic fuzzy metric space, then

(a) a sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for all $t > 0$ and $p > 0$

$$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1 \text{ and } \lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0$$

(b) a sequence $\{x_n\}$ in X is said to be convergent sequence if, for all $t > 0$ and $p > 0$

$$\lim_{n\to\infty} M(x_n, x, t) = 1 \text{ and } \lim_{n\to\infty} N(x_n, x, t) = 0$$

(c) An intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

(d) An intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ is said to be compact if

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every sequence in X contains a convergent subsequence.

Theorem 3.3.2[105]: Let (X, M, N, *, ◊) be a complete intuitionistic fuzzy metric space with continuous t-norm * *and* continuous t-conorm ◊ defined by $t * t \geq t$ and

$(1 - t) \lozenge (1 - t) \leq (1 - t)$ for all $t \in [0, 1]$ and $T: X \to CB(X)$ is multivalued map and

$\varphi: [0, \infty) \to [0, 1)$ is continuous function. There exists $0 < k < 1$ such that for all $x, y \in X$

$$M(Tx, Ty, kt) \geq M(x, y, t) * M(x, y, t)$$

and

$$N(Tx, Ty, kt) \leq N(x, y, t) \lozenge N(x, y, t),$$ then T has fixed point in X.

Proof: Let $\{x_n\}$ be a sequence in X and $x_{m-1} = x_m$ for some m, T has a fixed point x_m. Suppose that $x_{n-1} \neq x_n$

Then

$$M(x_n, x_{n+1}, kt) = M(Tx_{n+1}, Tx_{n+2}, kt)$$

$$\geq M(x_{n+1}, x_{n+2}, t/k) * M(x_{n+1}, x_{n+2}, t/k) \geq M(x_{n+1}, x_{n+2}, t/k^2)$$

And $$N(x_{n+1}, x_{n+2}, kt) \leq N(x_{n+1}, x_{n+2}, t/k^2)$$

Hence for any positive integer p

$$M(x_n, x_{n+p}, kt) \geq M(x_{n+1}, x_{n+2}, t/k) * \ldots\text{p-times}\ldots * M(x_{p+1-n}, x_{p+2-n}, t/k^n)$$

$$N(x_n, x_{n+p}, kt) \leq N(x_{n+1}, x_{n+2}, t/k) \lozenge \ldots\text{p-times}\ldots \lozenge N(x_{p+1-n}, x_{p+2-n}, t/k^n)$$

when $n \to \infty$ then

$$\lim_{n\to\infty} M(x_n, x_{n+1}, kt) \geq 1 * 1 * \ldots * 1 = 1$$ and

$$\lim_{n\to\infty} N(x_n, x_{n+1}, kt) \leq 0\Diamond\, 0 \,\Diamond \ldots \Diamond 0 = 0$$

It shows that $\{x_n\}$ is Cauchy sequence in X and so, by the completeness of X, $\{x_n\}$ converges to a point x, then

$$M(x_n, x, kt) \geq M(x_n, x, t/k^2)$$

and

$$N(x_n, x, kt) \leq N(x_n, x, t/k^2).$$

Let y be another fixed point in X and $x \neq y$ then

$$M(x_n, y, kt) = M(Tx_n, Ty, kt) \geq M(x_n, y, t/k^2) \text{ and}$$

$$N(x_n, y, kt) = N(Tx_n, Ty, kt) \leq N(x_n, y, t/k^2)$$

when $n \to \infty$ gives that $M(x_n, y, t/k^2) = 1$ and

$$N(x_n, y, t/k^2) = 0 \text{ for all } t > 0,$$

therefore it shows that $x = y$ so x is the fixed point of T.

Example Consider X = R with a Fuzzy metric. For A = [–1, 0] and B = [0, 1], $t > 0$, define F: $A^2 \cup B^2 \to A \cup B$ by $F(x, y) = -\frac{x+y}{4}$. Then F is a square cyclic contraction. By definition $F(A^2) \subseteq B$, $F(B^2) \subseteq A$ then $F(x, x') - F(y, y') = -\frac{x+y}{4} + \frac{y+y'}{4} \leq \frac{1}{4}(x-y) + (x'-y')$ for all $(x, x') \in A^2$ and $(y, y') \in B^2$. Moreover M(A, B, t) = 0 Then F has a unique coupled fixed point that is $(0, 0) \in A^2 \cup B^2$.

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